

Quantum mechanics on space with $SU(2)$ fuzziness

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Abstract

Quantum mechanics of models is considered which are constructed in spaces with Lie algebra type commutation relations between spatial coordinates. The case is specialized to that of the group $SU(2)$, for which the formulation of the problem via the Euler parameterization is also presented. $SU(2)$ -invariant systems are discussed, and the corresponding eigenvalue problem for the Hamiltonian is reduced to an ordinary differential equation, as it is the case with such models on commutative spaces.

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1 Introduction

In recent years there has been great interest to study physical models on spaces with noncommutative coordinates. In the simplest case of canonical noncommutative space the coordinates satisfy

$$[\hat{x}_\mu, \hat{x}_\nu] = i \theta_{\mu\nu} \mathbf{1}, \quad (1)$$

in which θ is an antisymmetric constant tensor and $\mathbf{1}$ represents the unit operator. It has been understood that the longitudinal directions of D-branes in the presence of a constant B-field background appear to be noncommutative, as seen by the ends of open strings [1–4]. The theoretical and phenomenological implications of such noncommutative coordinates have been extensively studied [5].

One direction to extend studies on noncommutative spaces is to consider spaces where the commutators of the coordinates are not constants. Examples of this kind are the noncommutative cylinder and the q -deformed plane [6], the so-called κ -Poincaré algebra [7–10], and linear noncommutativity of the Lie algebra type [11, 12]. In the latter the dimensionless spatial positions operators satisfy the commutation relations of a Lie algebra:

$$[\hat{x}_a, \hat{x}_b] = f^c{}_{ab} \hat{x}_c, \quad (2)$$

where $f^c{}_{ab}$'s are structure constants of a Lie algebra. One example of this kind is the algebra $SO(3)$, or $SU(2)$. A special case of this is the so called fuzzy sphere [13, 14], where an irreducible representation of the position operators is used which makes the Casimir of the algebra, $(\hat{x}_1)^2 + (\hat{x}_2)^2 + (\hat{x}_3)^2$, a multiple of the identity operator (a constant, hence the name sphere). One can consider the square root of this Casimir as the radius of the fuzzy sphere. This is, however, a noncommutative version of a two-dimensional space (sphere).

In [15–17] a model was introduced in which the representation was not restricted to an irreducible one, instead the whole group was employed. In particular the regular representation of the group was considered, which contains all representations. As a consequence in such models one is dealing with the whole space, rather than a sub-space, like the case of fuzzy sphere as a 2-dimensional surface. In [15] basic ingredients for calculus on a linear fuzzy space, as well as basic notions for a field theory on such a space, were introduced. In [16] basic elements for calculating the matrix elements corresponding to transition between initial and final states were discussed. Models based on the regular representation of $SU(2)$ were treated in more detail, giving explicit forms of the tools and notions introduced in their general forms [15, 16]. In [16] and [17] the tree and 1-loop diagrams for a self-interacting scalar field theory were discussed, respectively. It is observed that models based on Lie algebra type noncommutativity enjoy three features:

- They are free from any ultraviolet divergences if the group is compact.
- There is no momentum conservation in such theories.

- In the transition amplitudes only the so-called planar graphs contribute.

The reason for latter is that the non-planar graphs are proportional to δ -distributions whose dimensions are less than their analogues coming from the planar sector, and so their contributions vanish in the infinite-volume limit usually taken in transition amplitudes [17].

The facts that in such theories the mass-shell condition is different, and there is no momentum conservation, lead to different consequences (with respect to ordinary theories) in collisions. This was exploited in [18], where it was seen that there may be a new threshold for the collision of two massless particles to produce massive particles.

In [19] the classical mechanics defined on a space with $SU(2)$ fuzziness was studied. In particular, the Poisson structure induced by noncommutativity of $SU(2)$ type was investigated, for either Cartesian or Euler parameterization of $SU(2)$ group. The consequences of $SU(2)$ -symmetry in such spaces on integrability, was also studied in [19].

The purpose of the present work is to examine the quantum mechanics on a space with $SU(2)$ fuzziness. In particular quantum models are studied which have $SU(2)$ -symmetry.

The scheme of the rest of this paper is the following. In section 2, the commutation relations of the position and momentum operators corresponding to spaces with Lie-algebra noncommutativity in the configuration space are studied. In section 3, these are specialized to the group $SU(2)$. In section 4 systems are studied which are $SU(2)$ -invariant, and the eigen-value problem for the corresponding Hamiltonian is reduced to an ordinary differential equation.

2 The quantum commutators

Consider a Lie group G . Denote the members of a basis for the left-invariant vector fields corresponding to this group by \hat{x}_a 's. These fields satisfy (2), with the structure constants of the Lie algebra corresponding to G . The coordinates \hat{k}^a are defined such that

$$U(\hat{\mathbf{k}}) := [\exp(\hat{k}^a \hat{x}_a)] U(\mathbf{0}), \quad (3)$$

where $U(\hat{\mathbf{k}})$ is the group element corresponding to the coordinates $\hat{\mathbf{k}}$, $U(\mathbf{0})$ is the identity, and $\exp(\hat{x})$ is the flux corresponding to the vector field \hat{x} . The action of $L_{\hat{x}_a}$ (the Lie derivative corresponding to the vector field \hat{x}_a) on an arbitrary scalar function F can be written like

$$L_{\hat{x}_a}(F) = \hat{x}_a{}^b \frac{\partial F}{\partial \hat{k}^b}, \quad (4)$$

where $\hat{x}_a{}^b$'s are scalar functions, and satisfy

$$\hat{x}_a{}^b(\hat{\mathbf{k}} = \mathbf{0}) = \delta_a^b. \quad (5)$$

One can define the vector fields \hat{X}_a locally through

$$L_{\hat{X}_a}(F) = \frac{\partial F}{\partial \hat{k}^a}, \quad (6)$$

so that

$$\hat{x}_a = \hat{x}_a{}^b \hat{X}_b. \quad (7)$$

Then, considering scalar functions as operators acting on scalar functions through simple multiplications, and vector fields as operators acting on scalar functions through Lie derivation, one arrives at the following commutation relations.

$$[\hat{X}_a, \hat{X}_b] = 0, \quad (8)$$

$$[\hat{X}_a, \hat{k}^b] = \delta_a^b, \quad (9)$$

$$[\hat{k}^a, \hat{k}^b] = 0. \quad (10)$$

One should, however, remember that the functions \hat{k}^a and the vector fields \hat{X}_a are only locally defined. One can write the above commutation relations in terms of \hat{x}_a 's instead of \hat{X}_a 's. The equation corresponding to (8) would be (2), while that corresponding to (9) would be

$$[\hat{x}_a, \hat{k}^b] = \hat{x}_a{}^b, \quad (11)$$

and as $\hat{x}_a{}^b$'s are scalar functions, they commute with \hat{k}^a 's.

Next consider the right-invariant vector fields \hat{x}_a^R , so that they coincide with their left-invariant analogues at the identity of the group:

$$\hat{x}_a^R(\hat{\mathbf{k}} = \mathbf{0}) = \hat{x}_a(\hat{\mathbf{k}} = \mathbf{0}). \quad (12)$$

These field satisfy the commutation relations

$$[\hat{x}_a^R, \hat{x}_b^R] = -f^c{}_{ab} \hat{x}_c^R, \quad (13)$$

$$[\hat{x}_a^R, \hat{x}_b] = 0. \quad (14)$$

Using these, one defines the new vector field \hat{J}_a through

$$\hat{J}_a := \hat{x}_a - \hat{x}_a^R. \quad (15)$$

These are the generators of the adjoint action, and satisfy the commutation relations

$$[\hat{J}_a, \hat{J}_b] = f^c{}_{ab} \hat{J}_c, \quad (16)$$

$$[\hat{J}_a, \hat{X}_b] = f^c{}_{ab} \hat{X}_c, \quad (17)$$

$$[\hat{J}_a, \hat{x}_b] = f^c{}_{ab} \hat{x}_c, \quad (18)$$

$$[\hat{k}^c, \hat{J}_a] = f^c{}_{ab} \hat{k}^b. \quad (19)$$

Equations (9), (17), and (19) show that

$$\hat{J}_a = -f^c{}_{ab} \hat{k}^b \hat{X}_c. \quad (20)$$

Equation (9) ensures that there is no ambiguity in the order of \hat{k}^b and \hat{X}_c in the above.

Using the operators introduced in the above, one can easily construct the corresponding quantum operators. All one needs is to multiply these operators by suitable factors to make them Hermitian with proper dimension:

$$p^a := (\hbar/\ell) \hat{k}^a, \quad (21)$$

$$X_a := i \ell \hat{X}_a, \quad (22)$$

$$x_a := i \ell \hat{x}_a, \quad (23)$$

$$x_a{}^b(\mathbf{p}) := \hat{x}_a{}^b[(\ell/\hbar) \mathbf{p}], \quad (24)$$

$$J_a := i \hbar \hat{J}_a, \quad (25)$$

where ℓ is a constant of dimension length. One then arrives at the following commutation relations.

$$[p^a, p^b] = 0, \quad (26)$$

$$[X_a, p^b] = i \hbar \delta_a^b, \quad (27)$$

$$[X_a, X_b] = 0, \quad (28)$$

$$[x_a, p^b] = i \hbar x_a{}^b, \quad (29)$$

$$[x_a, x_b] = i \ell f^c{}_{ab} x_c, \quad (30)$$

$$[J_a, X_b] = i \hbar f^c{}_{ab} X_c, \quad (31)$$

$$[J_a, x_b] = i \hbar f^c{}_{ab} x_c, \quad (32)$$

$$[p^c, J_a] = i \hbar f^c{}_{ab} p^b, \quad (33)$$

$$[J_a, J_b] = i \hbar f^c{}_{ab} J_c, \quad (34)$$

Using (5), it is seen that in the limit $\ell \rightarrow 0$ the ordinary commutation relations are retrieved.

3 The group SU(2), and the Euler parameters

For the group SU(2), one also can define the Euler parameters through

$$[\exp(\phi T_3)] [\exp(\theta T_2)] [\exp(\psi T_3)] := [\exp(\hat{k}^a T_a)], \quad (35)$$

where T_a 's are the generators of SU(2) satisfying the commutation relation

$$[T_a, T_b] = \epsilon^c{}_{ab} T_c. \quad (36)$$

Using these, one arrives at

$$L_{\hat{x}_1}(F) = -\frac{\cos \psi}{\sin \theta} \frac{\partial F}{\partial \phi} + \sin \psi \frac{\partial F}{\partial \theta} + \frac{\cos \psi \cos \theta}{\sin \theta} \frac{\partial F}{\partial \psi}, \quad (37)$$

$$L_{\hat{x}_2}(F) = \frac{\sin \psi}{\sin \theta} \frac{\partial F}{\partial \phi} + \cos \psi \frac{\partial F}{\partial \theta} - \frac{\sin \psi \cos \theta}{\sin \theta} \frac{\partial F}{\partial \psi}, \quad (38)$$

$$L_{\hat{x}_3}(F) = \frac{\partial F}{\partial \psi}, \quad (39)$$

and

$$\begin{aligned} L_{\hat{j}_1}(F) &= \frac{\cos \phi \cos \theta - \cos \psi}{\sin \theta} \frac{\partial F}{\partial \phi} + (\sin \phi + \sin \psi) \frac{\partial F}{\partial \theta} \\ &\quad + \frac{-\cos \phi + \cos \psi \cos \theta}{\sin \theta} \frac{\partial F}{\partial \psi}, \end{aligned} \quad (40)$$

$$\begin{aligned} L_{\hat{j}_2}(F) &= \frac{\sin \phi \cos \theta + \sin \psi}{\sin \theta} \frac{\partial F}{\partial \phi} + (-\cos \phi + \cos \psi) \frac{\partial F}{\partial \theta} \\ &\quad + \frac{-\sin \phi - \sin \psi \cos \theta}{\sin \theta} \frac{\partial F}{\partial \psi}, \end{aligned} \quad (41)$$

$$L_{\hat{j}_3}(F) = -\frac{\partial F}{\partial \phi} + \frac{\partial F}{\partial \psi}, \quad (42)$$

for an arbitrary scalar field F . One also has

$$\cos \frac{\hat{k}}{2} = \cos \frac{\theta}{2} \cos \frac{\phi + \psi}{2}, \quad (43)$$

where

$$\hat{k} := \sqrt{\delta_{ab} \hat{k}^a \hat{k}^b}. \quad (44)$$

Euler parameterization is just an alternative parameterization of k_a 's as the momenta. Corresponding to these, one introduces the coordinate operators \hat{X}_ϕ , \hat{X}_θ , and \hat{X}_ψ . These satisfy

$$[\hat{X}_\alpha, \hat{k}^\beta] = \delta_\alpha^\beta, \quad (45)$$

where α and β are ϕ , θ , or ψ , and k^β has been defined as β itself. All other commutators vanish. The simplest realization for the above coordinate operators so that these operators are anti-Hermitian as well, is

$$\hat{X}_\alpha = \frac{1}{\sqrt{|\det \nu|}} \partial_\alpha \sqrt{|\det \nu|}, \quad (46)$$

where ν is the weight function appearing in the Haar measure $d\mu$:

$$d\mu = \nu d\phi d\theta d\psi. \quad (47)$$

Knowing that

$$\nu = c |\sin \theta|, \quad (48)$$

where c is a constant, it turns out that

$$\begin{aligned}\hat{X}_\phi &= \frac{\partial}{\partial\phi}, \\ \hat{X}_\theta &= \frac{1}{\sqrt{|\sin\theta|}} \frac{\partial}{\partial\theta} \sqrt{|\sin\theta|}, \\ \hat{X}_\psi &= \frac{\partial}{\partial\psi}.\end{aligned}\tag{49}$$

Then, one can use the differential operators in the right-hand sides of (37) to (42) as realizations of x_a 's and J_a 's, provided the following changes are performed on them. The changes are symmetrization with respect to Euler coordinates and their corresponding differential operators, using \hat{X}_α instead of ∂_α , and proper scaling so that the dimensions of the operators are correct and the operators are Hermitian. These result in the following realization

$$x_1 = i\ell \left(-\frac{\cos\psi}{\sin\theta} \frac{\partial}{\partial\phi} + \sin\psi \frac{\partial}{\partial\theta} + \frac{\cos\psi \cos\theta}{\sin\theta} \frac{\partial}{\partial\psi} \right),\tag{50}$$

$$x_2 = i\ell \left(\frac{\sin\psi}{\sin\theta} \frac{\partial}{\partial\phi} + \cos\psi \frac{\partial}{\partial\theta} - \frac{\sin\psi \cos\theta}{\sin\theta} \frac{\partial}{\partial\psi} \right),\tag{51}$$

$$x_3 = i\ell \frac{\partial}{\partial\psi},\tag{52}$$

$$\begin{aligned}J_1 = i\hbar \left[\frac{\cos\phi \cos\theta - \cos\psi}{\sin\theta} \frac{\partial}{\partial\phi} + (\sin\phi + \sin\psi) \frac{\partial}{\partial\theta} \right. \\ \left. + \frac{-\cos\phi + \cos\psi \cos\theta}{\sin\theta} \frac{\partial}{\partial\psi} \right],\end{aligned}\tag{53}$$

$$\begin{aligned}J_2 = i\hbar \left[\frac{\sin\phi \cos\theta + \sin\psi}{\sin\theta} \frac{\partial}{\partial\phi} + (-\cos\phi + \cos\psi) \frac{\partial}{\partial\theta} \right. \\ \left. + \frac{-\sin\phi - \sin\psi \cos\theta}{\sin\theta} \frac{\partial}{\partial\psi} \right],\end{aligned}\tag{54}$$

$$J_3 = i\hbar \left(-\frac{\partial}{\partial\phi} + \frac{\partial}{\partial\psi} \right),\tag{55}$$

Introducing the new parameters χ and ξ :

$$\chi := \frac{\phi - \psi}{2}\tag{56}$$

$$\xi := \frac{\phi + \psi}{2},\tag{57}$$

it is seen that

$$J_\pm = i\hbar \exp(\pm i\chi) \left(-\tan\frac{\theta}{2} \cos\xi \frac{\partial}{\partial\xi} + 2 \sin\xi \frac{\partial}{\partial\theta} \pm i \cot\frac{\theta}{2} \sin\xi \frac{\partial}{\partial\chi} \right)\tag{58}$$

$$J_3 = -i\hbar \frac{\partial}{\partial\chi},\tag{59}$$

where

$$J_{\pm} = J_1 \pm i J_2. \quad (60)$$

Again introducing new variables

$$v := \cos \frac{\theta}{2} \cos \xi, \quad (61)$$

$$\tau := (1 - v^2)^{-1/2} \cos \frac{\theta}{2} \sin \xi, \quad (62)$$

$$s^2 := 1 - \tau^2, \quad (63)$$

one arrives at

$$\begin{aligned} J_{\pm} &= i \hbar \exp(\pm i \chi) \left(-\sqrt{1 - \tau^2} \frac{\partial}{\partial \tau} \pm i \frac{\tau}{\sqrt{1 - \tau^2}} \frac{\partial}{\partial \chi} \right), \\ &= i \hbar \exp(\pm i \chi) \sqrt{1 - s^2} \left(\frac{\partial}{\partial s} \pm \frac{i}{s} \frac{\partial}{\partial \chi} \right), \end{aligned} \quad (64)$$

$$J_3 = -i \hbar \frac{\partial}{\partial \chi}, \quad (65)$$

resulting in

$$\begin{aligned} \mathbf{J} \cdot \mathbf{J} &= -\hbar^2 \left[(1 - \tau^2) \frac{\partial^2}{\partial \tau^2} - 2\tau \frac{\partial}{\partial \tau} + \frac{1}{1 - \tau^2} \frac{\partial^2}{\partial \chi^2} \right], \\ &= -\hbar^2 \left[(1 - s^2) \frac{\partial^2}{\partial s^2} + \frac{1 - 2s^2}{s} \frac{\partial}{\partial s} + \frac{1}{s^2} \frac{\partial^2}{\partial \chi^2} \right], \end{aligned} \quad (66)$$

where

$$\mathbf{A} \cdot \mathbf{B} := \delta_{ab} A^a B^b. \quad (67)$$

Using (65) and (66), it is seen that the angular momentum eigenfunctions (\mathcal{Y}_l^m 's) satisfying

$$\begin{aligned} J_3 \mathcal{Y}_l^m &= m \hbar \mathcal{Y}_l^m, \\ \mathbf{J} \cdot \mathbf{J} \mathcal{Y}_l^m &= l(l + 1) \hbar^2 \mathcal{Y}_l^m, \end{aligned} \quad (68)$$

are products of an arbitrary function f of v , and Y_l^m 's (the usual spherical harmonics) with the cosine of the colatitude equal to τ and the longitude equal to χ , that is

$$\mathcal{Y}_l^m = f(v) Y_l^m(\cos^{-1} \tau, \chi). \quad (69)$$

4 SU(2)-invariant quantum systems

Consider a configuration space with linear SU(2)-fuzziness, and the corresponding Hilbert space on which the momenta and coordinates introduced in section 3 act. A system characterized by a Hamiltonian H , is said to be SU(2)-invariant, if H is SU(2)-invariant, that is if the commutators of H with J_a 's vanish. A

Hamiltonian which is a function of only $(\mathbf{p} \cdot \mathbf{p})$ and $(\mathbf{x} \cdot \mathbf{x})$ is clearly so. The aim is to exploit the SU(2)-symmetry of such a Hamiltonian to write down an eigenvalue equation for the Hamiltonian so that that equation contains only one variable (from the three variables corresponding to the momentum). To do so, one calculates $(\mathbf{x} \cdot \mathbf{x})$. The result is

$$\mathbf{x} \cdot \mathbf{x} = -\ell^2 \left(\frac{1 + \cos \theta}{2 \sin^2 \theta} \frac{\partial^2}{\partial \chi^2} + \frac{1 - \cos \theta}{2 \sin^2 \theta} \frac{\partial^2}{\partial \xi^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} \right), \quad (70)$$

or

$$\begin{aligned} \mathbf{x} \cdot \mathbf{x} &= -\ell^2 \left\{ \frac{1}{4(1-v^2)} \left[(1-s^2) \frac{\partial^2}{\partial s^2} + \frac{1-2s^2}{s} \frac{\partial}{\partial s} + \frac{1}{s^2} \frac{\partial^2}{\partial \chi^2} \right] \right. \\ &\quad \left. + \frac{1-v^2}{4} \frac{\partial^2}{\partial v^2} - \frac{3v}{4} \frac{\partial}{\partial v} \right\}, \\ &= -\ell^2 \left[-\frac{\hbar^2}{4(1-v^2)} \mathbf{J} \cdot \mathbf{J} + \frac{1-v^2}{4} \frac{\partial^2}{\partial v^2} - \frac{3v}{4} \frac{\partial}{\partial v} \right]. \end{aligned} \quad (71)$$

From this, one finds that

$$\mathbf{x} \cdot \mathbf{x} \mathcal{Y}_l^m = \ell^2 Y_l^m \left[\frac{l(l+1)}{4(1-v^2)} - \frac{1-v^2}{4} \frac{d^2}{dv^2} + \frac{3v}{4} \frac{d}{dv} \right] f. \quad (72)$$

Substituting f in (69) with another function Υ ,

$$f(v) =: (1-v^2)^{l/2} \Upsilon(v), \quad (73)$$

one arrives at

$$\mathbf{x} \cdot \mathbf{x} \mathcal{Y}_l^m = \ell^2 (1-v^2)^{l/2} Y_l^m \left[-\frac{1-v^2}{4} \frac{d^2}{dv^2} + \left(\frac{3}{4} + \frac{l}{2} \right) v \frac{d}{dv} + \frac{l}{2} \left(\frac{l}{2} + 1 \right) \right] \Upsilon. \quad (74)$$

Now assume that the Hamiltonian is the sum of a kinetic term K , which is a function only $(\mathbf{p} \cdot \mathbf{p})$, and a potential term V , which is function of only $(\mathbf{x} \cdot \mathbf{x})$:

$$H = K + V. \quad (75)$$

Noting that $(\mathbf{p} \cdot \mathbf{p})$ is a function of only v , the eigenvalue equation for H becomes

$$K \Upsilon + V \left\{ \mathbf{x} \cdot \mathbf{x} = \ell^2 \left[-\frac{1-v^2}{4} \frac{d^2}{dv^2} + \left(\frac{3}{4} + \frac{l}{2} \right) v \frac{d}{dv} + \frac{l}{2} \left(\frac{l}{2} + 1 \right) \right] \right\} \Upsilon = E \Upsilon, \quad (76)$$

where K is a function of only v . An example for K is [16–19]

$$\begin{aligned} K &= \frac{4\hbar^2}{\ell^2 m} \left(1 - \cos \frac{\ell p}{2\hbar} \right), \\ &= \frac{4\hbar^2}{\ell^2 m} (1-v). \end{aligned} \quad (77)$$

One then arrives at

$$\begin{aligned} & \frac{4\hbar^2}{\ell^2 m} (1-v) \Upsilon \\ +V \left\{ \mathbf{x} \cdot \mathbf{x} = \ell^2 \left[-\frac{1-v^2}{4} \frac{d^2}{dv^2} + \left(\frac{3}{4} + \frac{l}{2} \right) v \frac{d}{dv} + \frac{l}{2} \left(\frac{l}{2} + 1 \right) \right] \right\} \Upsilon = E \Upsilon, \end{aligned} \quad (78)$$

If the potential function V is bounded from above, the eigenvalues of the Hamiltonian would be bounded from above, with the following as an upper bound

$$E \leq V_{\max} + \frac{8\hbar^2}{\ell^2 m}, \quad (79)$$

where V_{\max} is the maximum of V . If it is possible that V takes large values (compared to the maximum of K), then large eigenvalues are possible for the Hamiltonian and these correspond to eigenvectors which are approximately eigenvectors of $(\mathbf{x} \cdot \mathbf{x})$:

$$\begin{aligned} (\mathbf{x} \cdot \mathbf{x}) \Psi &= \ell^2 j(j+1) \Psi, \\ H \Psi &= E \Psi, \\ \frac{E}{V[\mathbf{x} \cdot \mathbf{x} = \ell^2 j(j+1)]} &\approx 1, \end{aligned} \quad (80)$$

where $(2j)$ is a nonnegative integer.

As examples consider the free particle and the harmonic oscillator. For the former there is no potential term, and the spectrum of the Hamiltonian is bounded from above (and of course below):

$$0 \leq E \leq \frac{8\hbar^2}{\ell^2 m}. \quad (81)$$

For the latter, one uses the potential

$$V = \frac{1}{2} m \omega^2 \mathbf{x} \cdot \mathbf{x}, \quad (82)$$

to arrive at the following equation for the eigenvalue problem.

$$\begin{aligned} & \left\{ \frac{4\hbar^2}{\ell^2 m} (1-v) \right. \\ & \left. + \frac{\ell^2}{2} m \omega^2 \left[-\frac{1-v^2}{4} \frac{d^2}{dv^2} + \left(\frac{3}{4} + \frac{l}{2} \right) v \frac{d}{dv} + \frac{l}{2} \left(\frac{l}{2} + 1 \right) \right] \right\} \Upsilon = E \Upsilon. \end{aligned} \quad (83)$$

As the corresponding potential is not bounded from above for large values of $(\mathbf{x} \cdot \mathbf{x})$, one can use the above argument to see that for large values of j , the eigenvectors of $(\mathbf{x} \cdot \mathbf{x})$ corresponding to the eigenvalue $\ell^2 j(j+1)$ are also eigenvalues of the Hamiltonian, and the corresponding energies satisfy

$$\lim_{j \rightarrow \infty} \frac{E_j}{j(j+1)} = \frac{1}{2} m \omega^2 \ell^2. \quad (84)$$

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